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Flow past a suddenly heated vertical plate

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An analysis is presented that is appropriate to three distinct phases in the temporal development of the flow past a semi-infinite vertical plate whose temperature is suddenly changed from that of the surrounding fluid. These are the initial stage when a one-dimensional solution describes the flow, then a local solution which describes the early stages of the departure from this, and finally an asymptotic solution which describes the manner in which the final steady state is achieved.

1. Introduction

The steady free convective flow from a heated vertical plate was first examined by Lorenz (1881). The work of Schmidt & Beckmann (1930) showed that the assumptions made by Lorenz were invalid, and their definitive paper, which includes both an experimental and a theoretical investigation, forms the basis (see Ostrach 1964) for studies of the classical problem of steady free-convection boundary-layer flow past a semi-infinite vertical plate. In this paper we consider certain features associated with the development of this steady flow from a state of rest.

The unsteady flow which arises when the temperature of a semi-infinite vertical plate is suddenly raised to a uniform and constant value which exceeds that of the ambient fluid may be qualitatively described as follows. At a finite distance from the leading edge, the flow initially develops as if the plate were infinite in extent. Owing to the wavelike nature of the unsteady boundary-layer equations, a finite time elapses before the leading edge influences the flow development at that station; thereafter a transition to the classical steady-state solution takes place. This description of the flow was first given by Siegel (1958) and applies equally to the case when the flow is induced by introducing more general thermal conditions on the plate at the initial instant. Siegel, using an approximate method based upon integrated forms of the governing equations, analysed the situations in which there is a sudden change in temperature or a sudden change in heat flux at the plate.

The flows under discussion have been the subject of experimental investigations, and since density gradients are an essential feature of these flows interferometric techniques have been employed to visualize them. Published interferograms by Goldstein & Eckert (1960) and by Gebhart, Dring & Polymeropoulos (1967) confirm the overall features of the unsteady free-convection flows described above. More detailed comparison by the former authors between experiment and the theory of Siegel shows qualitative agreement. In a series of papers Gebhart (1961, 1963*a*, *b*, 1964) uses approximate methods to analyse the flow development for a variety of initial conditions, and emphasizes the importance of a physically realistic surface condition. Details of the analysis and of its agreement with experiment are readily accessible in Gebhart (1972), a paper that includes a discussion of the leading-edge effect.

In the present paper we consider the unsteady flow which arises following a step change in wall temperature. This idealized model enables us to reduce the number of independent variables from three to two, and therefore has the limitations of any similarity solution. At present a complete analysis even of this idealized problem is not possible. Our aim, as a step towards such an analysis, is to examine certain key features which we expect to be essential to a complete understanding of the problem. Specifically, we examine in § 2 the initial phase, in § 3 the initiation of the transition from this initial phase and in § 4 the final decay to the classical steady-state flow. The initial development of the flow which, as already indicated, corresponds to the flow past a suddenly heated infinite plane has been given previously by Illingworth (1950). The manner in which the flow departs from this initial one-dimensional form has certain features in common with the flow past a semi-infinite plate which is set in motion impulsively in its own plane. That problem has been analysed by Stewartson (1951, 1973) and numerical solutions of it have been obtained by Hall (1969) and Dennis (1972). The analytical and numerical results, where comparable, are in accord. In that problem, also, departures from the one-dimensional, or Rayleigh, solution commence at any station after a finite time. The minimum time which elapses before such departures take place is the time taken for a signal to travel from the leading edge. It can be shown that the influence of the leading edge is first brought to a given station by the signal that is convected along the boundary layer by the fluid moving fastest, relative to the plate, at each station. For the impulsively started plate, the fastest relative speed in the boundary layer is the free-stream speed, i.e. the constant speed of the plate itself. For the problem under consideration we apply the same arguments to calculate the critical time at which departures from the one-dimensional solution first begin, and we note that the maximum relative speed occurs inside the boundary layer and is time-dependent. The difference between this argument and that of Goldstein & Briggs (1964) will be explained in § 2.

In the present problem a similarity reduction of the governing boundary-layer equations is available and at the critical value τ_0 of the time-like variable τ, defined below, departures from the one-dimensional solution manifest themselves through an essential singularity. The local analysis which we employ in the neighbourhood of $\tau = \tau_0$ leads to an eigenvalue problem, and the departures from the one-dimensional solution are undetermined to within a multiplicative constant. This is to be expected since in the similarity formulation disturbances can propagate, for $\tau > \tau_0$, in the direction of either positive *or* negative τ. Thus the indeterminacy arises since no conditions at $\tau = \infty$, which for any finite time corresponds to the leading edge of the plate, are imposed upon our local solution. Likewise for $\tau \gg 1$, where we examine the final decay to the steady state, a perturbation analysis yields an eigenvalue problem, and disturbances to the steady-state solution remain undetermined to within an arbitrary constant. In this case, of course, the indeterminacy arises since the conditions at $\tau = 0$ are not invoked. The analysis for $\tau \approx \tau_0$ and $\tau \gg 1$, although significantly different in detail, is carried out in the same spirit as that given by Stewartson (1973) and Watson (in an appendix to Dennis 1972) for the impulsively moved plate. We conclude this section by deriving the similarity form of the governing equations.

In our mathematical description of the problem we use the boundary-layer equations and assume that our fluid is a Boussinesq fluid. We choose $(\bar{x}, \bar{y})$ as co-ordinates along and normal to the plate with $(\partial\bar{\psi}/\partial\bar{y}, -\partial\bar{\psi}/\partial\bar{x})$ as the corresponding components of velocity. The origin of co-ordinates is assumed to be on the leading edge of the plate and the gravitational field $\mathbf{g}$ is equal to $-g\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the $\bar{x}$ direction. The temperature T takes the values T_w and T_∞ at the wall and in the ambient fluid respectively, and we assume that $T_w > T_\infty$. A characteristic velocity for this free-convection situation is

$$U = [g\beta L(T_w - T_\infty)]^{\frac{1}{2}},$$

where β is the volumetric coefficient of thermal expansion and L a typical length. If we define the Grashof number $Gr = \beta g L^3 (T_w - T_\infty)/\nu^2$, where ν is the kinematic viscosity of the fluid, then our use of the boundary-layer equations is only justified if $Gr \gg 1$. Following Ostrach (1964) we introduce dimensionless variables appropriate to this situation as

$$x = \bar{x}/L, \quad y = Gr^{\frac{1}{4}}\bar{y}/L, \quad t = U\bar{t}/L,$$
$$\psi = Gr^{\frac{1}{4}}\bar{\psi}/UL, \quad T = T_\infty + (T_w - T_\infty)\,\theta,$$

where $\bar{t}$ denotes time. If K is the thermal diffusivity of the fluid then the Prandtl number σ is equal to ν/K.

Under the assumptions we have made, the governing equations may be reduced to similarity form by writing

$$\psi = x^{\frac{3}{4}} f(\eta, \tau), \quad \theta = \theta(\eta, \tau), \tag{1}$$

where

$$\eta = y/x^{\frac{1}{4}}, \quad \tau = t/x^{\frac{1}{2}}, \tag{2}$$

and f and θ satisfy the equations

$$\left.\begin{aligned} \left(1 - \tfrac{1}{2}\tau\frac{\partial f}{\partial \eta}\right)\frac{\partial^2 f}{\partial \tau\,\partial \eta} + \left(\tfrac{1}{2}\tau\frac{\partial f}{\partial \tau} - \tfrac{3}{4}f\right)\frac{\partial^2 f}{\partial \eta^2} + \frac{1}{2}\left(\frac{\partial f}{\partial \eta}\right)^2 &= \theta + \frac{\partial^3 f}{\partial \eta^3}, \\ \left(1 - \tfrac{1}{2}\tau\frac{\partial f}{\partial \eta}\right)\frac{\partial \theta}{\partial \tau} + \left(\tfrac{1}{2}\tau\frac{\partial f}{\partial \tau} - \tfrac{3}{4}f\right)\frac{\partial \theta}{\partial \eta} &= \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial \eta^2}, \end{aligned}\right\} \tag{3}$$

together with

$$\left.\begin{array}{llll} f = \partial f/\partial \eta = 0 & \text{at} & \eta = 0 & \text{for all } \tau, \\ \theta, \partial f/\partial \eta \to 0 & \text{as} & \eta \to \infty & \text{for all } \tau, \\ \theta = 1 & \text{on} & \eta = 0 & \text{for} \quad \tau > 0, \\ \theta = \partial f/\partial \eta = 0 & \text{at} & \tau = 0 & \text{for} \quad \eta > 0. \end{array}\right\} \tag{4}$$

If in (3) we neglect derivatives with respect to τ and write $f(\eta,\tau) = f_0(\eta)$ and $\theta(\eta,\tau) = \theta_0(\eta)$ then we recover the classical steady-state equations for free-convection flow over a semi-infinite flat plate discussed at the beginning of this section, namely

$$\left.\begin{aligned} f_0''' + \tfrac{3}{4}f_0 f_0'' - \tfrac{1}{2}f_0'^2 + \theta_0 &= 0,\\ \theta_0'' + \tfrac{3}{4}\sigma f_0 \theta_0' &= 0,\end{aligned}\right\} \tag{5}$$

for which $$f_0(0) = f_0'(0) = 0, \quad \theta_0(0) = 1, \quad f_0'(\infty) = \theta_0(\infty) = 0. \tag{6}$$

For simplicity we henceforth restrict our attention to $\sigma = 1$. If $\sigma \neq 1$ the conclusions are similar but the analysis is more complicated.

2. The initial development: $\tau < \tau_0$

As with any impulsive boundary-layer flow, which is realized in the present case by suddenly giving the plate at $t = 0$ a finite and constant temperature exceeding that of its surroundings, there is an initiation period in which $y/\sqrt{t}$, which is equal to $\eta/\sqrt{\tau}$, is an appropriate independent variable. Accordingly for the initial development of the solution we find it convenient to use independent variables (ζ, τ), where

$$\zeta = \eta/2\sqrt{\tau}, \tag{7}$$

and we write $$f(\eta,\tau) = 2\tau^{\frac{3}{2}}F(\zeta,\tau), \quad \theta(\eta,\tau) = \Theta(\zeta,\tau). \tag{8}$$

It follows from (3) that the dependent variables F and Θ satisfy the equations

$$\left.\begin{aligned} \tau\left(1 - \tfrac{1}{2}\tau^2\frac{\partial F}{\partial \zeta}\right)\frac{\partial^2 F}{\partial \tau\,\partial \zeta} + \tfrac{1}{2}\tau^3\frac{\partial^2 F}{\partial \zeta^2}\frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial \zeta} - \tfrac{1}{2}\zeta\frac{\partial^2 F}{\partial \zeta^2} &= \Theta + \frac{1}{4}\frac{\partial^3 F}{\partial \zeta^3},\\ \tau\left(1 - \tfrac{1}{2}\tau^2\frac{\partial F}{\partial \zeta}\right)\frac{\partial \Theta}{\partial \tau} + \tfrac{1}{2}\tau^3\frac{\partial \Theta}{\partial \zeta}\frac{\partial F}{\partial \tau} &= \frac{1}{4}\frac{\partial^2 \Theta}{\partial \zeta^2} + \tfrac{1}{2}\zeta\frac{\partial \Theta}{\partial \zeta}.\end{aligned}\right\} \tag{9}$$

If in (9) we neglect derivatives with respect to τ and write $F(\zeta,\tau) = F_0(\zeta)$ and $\Theta(\zeta,\tau) = \Theta_0(\zeta)$ we recover the classical one-dimensional equations which, subject to

$$\left.\begin{aligned} \Theta_0(0) = 1, \quad \Theta_0(\infty) &= 0,\\ F_0'(0) = F_0'(\infty) &= 0,\end{aligned}\right\}$$

admit the solution

$$F_0'(\zeta) = 2\zeta(e^{-\zeta^2}/\pi^{\frac{1}{2}} - \zeta\,\mathrm{erfc}\,\zeta), \quad \Theta_0 = 1 - \mathrm{erf}\,\zeta. \tag{10}$$

The corresponding solution with $\sigma \neq 1$ was also given by Illingworth (1950). The solution (10) describes, exactly, the flow past an infinite plate whose temperature is raised instantaneously to a value above the ambient temperature. It can, at most, describe only the initial phase of development of the flow past a *semi-infinite* plate since it does not satisfy the boundary conditions at $x = 0$, and thus contains no information about the leading edge. This is analogous to the situation in the problem of the impulsively started flat plate studied analytically by Stewartson (1951, 1973) and numerically by Hall (1969) and Dennis (1972). In his earlier paper Stewartson showed that the one-dimensional, or Rayleigh, solution is appropriate until, at any station, the boundary layer becomes aware of the existence of the leading edge. The time at which this occurs is calculated on the basis that the fastest moving disturbances are transmitted with the free-stream velocity at the edge of the boundary layer, and then diffuse instantaneously across this layer. In his later paper Stewartson (1973) found the eigensolutions which describe the departure of the flow from that given by the Rayleigh solution, and compared the skin friction and displacement thickness calculated from the first eigensolution with the numerical results of Dennis. The situation in the present problem is similar, and we also argue that the fastest leading-edge signal travels with the maximum speed in the boundary layer, which in this case occurs at an interior point. It follows then, from (1) and (8), that the time which elapses before a given station along the plate first becomes aware of the presence of the leading edge is given by

$$\tau = \tau_0 = [2/F'_{0\,\max}]^{\frac{1}{2}}, \tag{11}$$

and for $\tau < \tau_0$ the appropriate solution is that of (10).

Equation (11) can also be regarded as giving the maximum penetration distance of the leading-edge signal at any fixed time. It leads to a slightly greater maximum penetration distance than does the argument of Goldstein & Briggs (1964). Essentially they write for this distance

$$x_p = \max\left[\int_0^t u(y,t_1)\,dt_1\right]$$

whilst we write

$$x_p = \int_0^t \max\,[u(y,t_1)]\,dt_1,$$

where in both cases the maximization is with respect to y. The latter assumption, which leads to (11), is reinforced by an examination of equations (9). They are parabolic in nature and whether disturbances propagate in the direction of increasing or decreasing τ is expected to be determined by the sign of $1-\frac{1}{2}\tau^2\,\partial F/\partial\zeta$. Since, for $\tau < \tau_0$, we have $\partial F/\partial\zeta = F'_0$ it follows that at $\tau = \tau_0$ this coefficient vanishes at one point in the boundary layer. Thus to extend the solution to larger values of τ it is necessary to take account of the boundary conditions at $\tau = \infty$, since propagation in both directions is now possible. Since $\tau = \infty$ corresponds to $x = 0$ when t is non-zero, we see the justification for the interpretation of $x^{\frac{1}{2}}\tau_0$ as the time at which the leading-edge signal reaches x.

A possible solution of (9) is of course $F(\zeta,\tau) = F_0(\zeta)$, $\theta(\zeta,\tau) = \Theta_0(\zeta)$ for all τ. This solution cannot be unique, however, since it implies that the coefficient $1-\frac{1}{2}\tau^2\,\partial F/\partial\zeta$ is negative for some ζ for all $\tau > \tau_0$, and this enables boundary conditions to be imposed at $\tau = \infty$. The non-uniqueness is accounted for by eigensolutions which first make their appearance at $\tau = \tau_0$ and are the means by which the leading-edge makes its presence felt.

These eigensolutions, whose coefficients are determined by the boundary conditions at $\tau = \infty$, are similar to those found by Stewartson (1973) for the impulsive plate problem. They exhibit an essential singularity at $\tau = \tau_0$, at which point all derivatives are zero. This is as it should be since for $\tau < \tau_0$ the solution is independent of x. Stewartson found it necessary to divide the boundary layer into four separate regions, but in the present problem the configuration is less complicated and we require only three.

3. The transition period: $0 < \tau - \tau_0 \ll 1$

To determine the analytic form of the eigensolutions required for the stage where the flow starts to depart from the one-dimensional solution given by (10), we write

$$F(\zeta,\tau) = F_0(\zeta) + F_1(\zeta,\tau), \quad \theta(\zeta,\tau) = \Theta_0(\zeta) + \Theta_1(\zeta,\tau), \tag{12}$$

in (9), and assume that $|F_1| \ll |F_0|$ and $|\Theta_1| \ll |\Theta_0|$ in the transition stage so that products of these small perturbations may be ignored. We thus obtain

$$\tau(1-\tfrac{1}{2}\tau^2 F_0')\frac{\partial^2 F_1}{\partial\zeta\,\partial\tau} + \tfrac{1}{2}\tau^3 F_0''\frac{\partial F_1}{\partial\tau} + \frac{\partial F_1}{\partial\zeta} - \tfrac{1}{2}\zeta\frac{\partial^2 F_1}{\partial\zeta^2} = \Theta_1 + \frac{1}{4}\frac{\partial^3 F_1}{\partial\zeta^3}, \tag{13}$$

$$\tau(1-\tfrac{1}{2}\tau^2 F_0')\frac{\partial\Theta_1}{\partial\tau} + \tfrac{1}{2}\tau^3\Theta_0'\frac{\partial F_1}{\partial\tau} = \frac{1}{4}\frac{\partial^2\Theta_1}{\partial\zeta^2} + \tfrac{1}{2}\zeta\frac{\partial\Theta_1}{\partial\zeta}, \tag{14}$$

and solutions of (13) and (14) are required such that

$$\left.\begin{array}{l} F_1, \Theta_1 \to 0 \quad \text{as} \quad \tau\to\tau_0; \\ F_1 = \Theta_1 = \partial F_1/\partial\zeta = 0 \quad \text{on} \quad \zeta = 0; \qquad \Theta_1, \partial F_1/\partial\zeta \to 0 \quad \text{as} \quad \zeta\to\infty. \end{array}\right\} \tag{15}$$

It will emerge that to solve (13) and (14) subject to (15) in the neighbourhood of $\tau = \tau_0$ it is necessary to divide the boundary layer into three separate regions. The main region, or middle layer, is centred on the point $\zeta = \zeta_0$, where $F_0'(\zeta)$ has its maximum, and is of thickness $O[(\tau-\tau_0)^{\frac{1}{2}}]$. This region determines the eigenvalues and the form of the essential singularity, and the inner and outer layers are required merely to adjust the solution so that it can satisfy the boundary conditions at $\zeta = 0$ and at $\zeta = \infty$. The importance of this middle layer is not unexpected in view of the argument presented earlier. Physically it is a region in which diffusive effects are of prime importance in the response, by the boundary layer, to the arrival of the leading-edge signal.

The middle layer

It follows from (10) that, near $\zeta = \zeta_0$,

$$\left.\begin{array}{l} F_0'(\zeta) = A_0 - A_1(\zeta-\zeta_0)^2 - A_2(\zeta-\zeta_0)^3 + O[(\zeta-\zeta_0)^4] \quad (A_1 > 0), \\ \Theta_0'(\zeta) = -\alpha_0 - \alpha_1(\zeta-\zeta_0) + O[(\zeta-\zeta_0)^2], \end{array}\right\} \tag{16}$$

where the constants A_i and α_i are such that

$$\tau_0^2 A_0 = 2, \quad \alpha_1 = -2\zeta_0\alpha_0, \quad A_2 = -\tfrac{2}{3}(\alpha_0 + \zeta_0 A_1), \tag{17}$$

and their numerical values when required are calculated from (10) and (11). Thus with

$$\tau = \tau_0 + T, \quad \zeta = \zeta_0 + z, \tag{18}$$

the leading terms of $\tau(1-\frac{1}{2}\tau^2 F_0')$ are $\frac{1}{2}A_1\tau_0^3 z^2 - 2T$, which indicates that since $0 < T \ll 1$ the appropriate variables are T and Y, where

$$Y = z/T^{\frac{1}{2}}. \tag{19}$$

We now write

$$\left.\begin{aligned} F_1(\zeta,\tau) &= T^{\gamma}\exp\left[-\frac{c}{T}-\frac{d}{T^{\frac{1}{2}}}\right]\{\tilde{G}_0(Y)+T^{\frac{1}{2}}\tilde{G}_1(Y)+O(T)\},\\ \Theta_1(\zeta,\tau) &= T^{\delta}\exp\left[-\frac{c}{T}-\frac{d}{T^{\frac{1}{2}}}\right]\{\tilde{H}_0(Y)+T^{\frac{1}{2}}\tilde{H}_1(Y)+O(T)\},\end{aligned}\right\} \tag{20}$$

where γ, δ, c and d ($c > 0$) are constants to be determined, and retain the leading terms in (13) and (14). This gives

$$\tfrac{1}{4}\tilde{G}_0'''' - c(\tfrac{1}{2}A_1\tau_0^3Y^2-2)\tilde{G}_0' + cA_1\tau_0^3Y\tilde{G}_0 = -T^{\delta-\gamma+\frac{3}{2}}\tilde{H}_0, \tag{21}$$

$$\tfrac{1}{4}\tilde{H}_0'' - c(\tfrac{1}{2}A_1\tau_0^3Y^2-2)\tilde{H}_0 = -\tfrac{1}{2}c\alpha_0\tau_0^3T^{\gamma-\delta-1}\tilde{G}_0. \tag{22}$$

Since the terms on the right-hand sides of (21) and (22) must not be large compared with those on the left-hand sides of the corresponding equations, it is necessary that $\delta+1 \leqslant \gamma \leqslant \delta+\frac{3}{2}$. If $\gamma = \delta+1$ the right-hand side of (21) is negligible and so $\tilde{G}_0$ can be determined without reference to (22), and the equation to be solved for $\tilde{H}_0$ is then $\frac{1}{4}\tilde{H}_0''-c(\frac{1}{2}A_1\tau_0^3Y^2-2)\tilde{H}_0 = -\frac{1}{2}c\alpha_0\tau_0^3\tilde{G}_0$. If $\gamma = \delta+\frac{3}{2}$, equation (22) yields a homogeneous equation for $\tilde{H}_0$ and (21) a non-homogeneous equation for $\tilde{G}_0$ which has the term $-\tilde{H}_0$ on its right-hand side. If $\delta+1 < \gamma < \delta+\frac{3}{2}$ equations (21) and (22) yield homogeneous equations for $\tilde{G}_0$ and $\tilde{H}_0$ respectively. From these various possibilities, that which determines the leading eigenfunction will be the one which assigns the lowest possible value to c. To investigate the possibilities we first consider the homogeneous equations, which with

$$Y = bZ, \quad \tilde{G}_0(Y) = G_0(Z), \quad \tilde{H}_0(Y) = H_0(Z), \tag{23}$$

where

$$b^4 = (8cA_1\tau_0^3)^{-1}, \quad a = 8cb^2, \tag{24}$$

become

$$G_0''' - (\tfrac{1}{4}Z^2-a)G_0' + \tfrac{1}{2}ZG_0 = 0, \tag{25}$$

$$H_0'' - (\tfrac{1}{4}Z^2-a)H_0 = 0. \tag{26}$$

The solutions of (26), the parabolic cylinder functions, are exponentially large either as $Z\to-\infty$ or as $Z\to\infty$ unless $a = \frac{1}{2}, \frac{3}{2}, \ldots$, in which cases there is one solution of the form $P_a(Z)\exp(-\frac{1}{4}Z^2)$, where P_a is a polynomial. It can be shown that solutions of (25) satisfy

$$\frac{d}{dZ}\left(\frac{1}{Z}\frac{dG_0}{dZ}\right) = 4Z\exp(-\tfrac{1}{4}Z^2)\,w(\tfrac{1}{2}Z^2), \tag{27}$$

where $w(s)$ satisfies the confluent hypergeometric equation

$$sw'' + (\tfrac{5}{2}-s)w' + (\tfrac{1}{2}a-\tfrac{5}{4})w = 0. \tag{28}$$

We deduce therefore that the solutions of (27) are Z^2-4a together with two functions both of which are exponentially large either as $Z\to-\infty$ or as $Z\to\infty$ unless $a = -\frac{1}{2}, \frac{3}{2}, \ldots$, or $a = \frac{5}{2}, \frac{9}{2}, \ldots$. The necessity to match the solution with a solution in the inner layer precludes the possibility that $G_0(Z) \propto (Z^2-4a)$; consequently a is not unrestricted. The negative value of a in the above sequences is unacceptable since we have assumed that $a > 0$ in the derivation of (25) and (26).

Thus the lowest possible value of a is $\frac{1}{2}$, with $\gamma = \delta + \frac{3}{2}$; this gives

$$H_0(Z) = e^{-\frac{1}{4}Z^2} \tag{29}$$

and

$$G_0''' - (\tfrac{1}{4}Z^2 - \tfrac{1}{2})G_0' + \tfrac{1}{2}ZG_0 = -4b^3 e^{-\frac{1}{4}Z^2}. \tag{30}$$

The complementary functions of (30) are $Z^2 - 2$ and two functions which are exponentially large either as $Z \to -\infty$ or as $Z \to \infty$. The solution of (30) we shall take is

$$G_0(Z) = -2\pi^{\frac{1}{2}}b^3[\pi^{-\frac{1}{2}}Z\,e^{-\frac{1}{4}Z^2} + (\tfrac{1}{2}Z^2 - 1)\{\operatorname{erf}(\tfrac{1}{2}Z) + 1\}], \tag{31}$$

which has the asymptotic forms

$$G_0(Z) \sim -(8b^3/Z)\,e^{-\frac{1}{4}Z^2} \quad \text{as} \quad Z \to -\infty; \quad G_0(Z) \sim -2\pi^{\frac{1}{2}}b^3Z^2 \quad \text{as} \quad Z \to +\infty. \tag{32}$$

This solution therefore decays exponentially towards the wall, but increases algebraically at the outer edge of the region under consideration. A reversal of this behaviour can be realized by the addition of a suitable multiple of $Z^2 - 2$, but it emerges that it is then not possible to match the solution with that appropriate to the inner layer.

Since $a = \frac{1}{2}$ it follows from (24) that

$$c = \tfrac{1}{32}\tau_0^3 A_1 = 0{\cdot}656, \tag{33}$$

and to find the value of γ we return to (20) and calculate $\tilde{G}_1(Y)$, $\tilde{H}_1(Y)$ and $\tilde{H}_2(Y)$. The results are that $d = 0$ and

$$\gamma = 2 + \tfrac{5}{4}\zeta_0^2 + \tfrac{9}{4}\zeta_0\frac{A_2}{A_1} + \frac{61}{64}\left(\frac{A_2}{A_1}\right)^2 = 2{\cdot}452, \tag{34}$$

with ζ_0, A_2 and A_1 defined in (17).

Although the solutions (29) and (31) are exponentially small as $Z \to -\infty$ they do not in fact satisfy the boundary conditions on the wall. Nor does (31) allow the condition $\partial F_1/\partial\zeta \to 0$ as $\zeta \to \infty$ to be satisfied. Thus further solutions valid in inner and outer layers are required, and we first consider the outer layer in which $\zeta = O(1)$.

The outer layer

In the outer layer we write

$$F_1(\zeta, \tau) \approx T^{\gamma-1}e^{-c/T}\mathscr{G}(\zeta), \quad \Theta_1(\zeta, \tau) \approx T^{\lambda}e^{-c/T}\mathscr{H}(\zeta), \tag{35}$$

where the exponent $\gamma - 1$ in the expression for F_1 follows from (32) in conjunction with (20), and λ, which is to be determined, must be such that $\lambda > \gamma - \frac{3}{2}$ as $H_0(Z)$ given by (29) is exponentially small as $Z \to \infty$. Then near $\tau = \tau_0$ the leading terms of (13) and (14) are

$$c\tau_0(1 - \tfrac{1}{2}\tau_0^2F_0')\mathscr{G}' + \tfrac{1}{2}c\tau_0^3F_0''\mathscr{G} = T^{\lambda-\gamma+3}\mathscr{H}, \tag{36}$$

$$c\tau_0(1 - \tfrac{1}{2}\tau_0^2F_0')T^{\lambda-\gamma+1}\mathscr{H} + \tfrac{1}{2}c\tau_0^3\mathscr{G} = 0. \tag{37}$$

Thus, to avoid a contradiction, we require $\lambda = \gamma - 1$ and it then follows that

$$\mathscr{G}(\zeta) = B(1 - \tfrac{1}{2}\tau_0^2F_0'), \quad \mathscr{H}(\zeta) = -\tfrac{1}{2}B\tau_0^2\Theta_0', \tag{38}$$

where B is a constant to be determined, and the boundary conditions as $\zeta \to \infty$ are now satisfied. From (38) we have, near $\zeta = \zeta_0$,

$$\mathscr{G}(\zeta) = \tfrac{1}{2}BA_1\tau_0^2(\zeta - \zeta_0)^2 + O[(\zeta - \zeta_0)^3], \quad \mathscr{H}(\zeta) = \tfrac{1}{2}B\tau_0^2\alpha_0 + O[(\zeta - \zeta_0)], \tag{39}$$

and by matching with (31) the unknown constant B in (38) is determined from

$$\tfrac{1}{2}BA_1\tau_0^2 = -2\pi^{\frac{1}{2}}b. \tag{40}$$

It has been verified that with this choice for B the term $\mathscr{H}(\zeta)$ in (38) matches with the term $\tilde{H}_1(Y)$ which arises in the solution for Θ_1 in the middle layer.

The inner layer

In the inner layer $\zeta = O(1)$ but $\partial F_1/\partial\zeta \gg F_1$ and similarly for Θ_1. In (13) and (14) we write, for $T \ll 1$,

$$\partial F_1/\partial\zeta \approx T^{\gamma-\frac{1}{2}}Q(\zeta)\,e^{-P(\zeta)/T}, \quad \Theta_1(\zeta,\tau) \approx T^{\gamma-\frac{3}{2}}R(\zeta)\,e^{-P(\zeta)/T}, \tag{41}$$

where we have anticipated the powers of T required to match with the solution (20), (29), (31) of the middle layer. For $F_1(\zeta,\tau)$ we find it sufficient to use the asymptotic relation

$$F_1(\zeta,\tau) \sim -\frac{T}{P'(\zeta)}\frac{\partial F_1}{\partial\zeta}. \tag{42}$$

Equating the terms of orders $T^{\gamma-\frac{5}{2}}$ and $T^{\gamma-\frac{7}{2}}$ in (13) and (14) respectively we obtain, from each equation,

$$\tau_0(1-\tfrac{1}{2}\tau_0^2F_0')P = \tfrac{1}{4}P'^2, \tag{43}$$

whilst the terms of orders $T^{\gamma-\frac{3}{2}}$ and $T^{\gamma-\frac{5}{2}}$ lead to

$$\begin{aligned}\tau_0(1-\tfrac{1}{2}\tau_0^2F_0')(\gamma-\tfrac{1}{2})Q+(1-\tfrac{3}{2}\tau_0^2F_0')PQ-\tfrac{1}{2}\tau_0^3F_0''(PQ/P')&\\ +\tfrac{1}{2}\zeta P'Q+\tfrac{1}{4}[(P'Q)'+P'Q'] &= R\end{aligned} \tag{44}$$

and

$$\begin{aligned}\tau_0(1-\tfrac{1}{2}\tau_0^2F_0')(\gamma-\tfrac{3}{2})R+(1-\tfrac{3}{2}\tau_0^2F_0')PR+\tfrac{1}{2}\zeta P'R&\\ +\tfrac{1}{4}[(P'R)'+P'R'] &= 0.\end{aligned} \tag{45}$$

There are two solutions $P_1(\zeta)$ and $P_2(\zeta)$ of (43) with the same value at $\zeta = 0$, and to effect a match with (20), (29) and (31) as $\zeta\to\zeta_0$ and $Z\to-\infty$ we take

$$P(\zeta) = P_1(\zeta),$$

where

$$P_1(\zeta) = \left[\int_\zeta^{\zeta_0}\tau_0^{\frac{1}{2}}[1-\tfrac{1}{2}\tau_0^2F_0'(\zeta_1)]^{\frac{1}{2}}\,d\zeta_1+c^{\frac{1}{2}}\right]^2. \tag{46}$$

Since it follows from (16) and (24) that near $\zeta = \zeta_0$

$$P_1(\zeta) = c+[(\zeta-\zeta_0)^2/4b^2]+O[(\zeta-\zeta_0)^3], \tag{47}$$

we see that both terms, $-c/T-\frac{1}{4}Z^2$, in the exponent of (20), with H_0 as in (29) and G_0 as in (31), are matched at this stage.

The corresponding solutions of (44) and (45) which enable us to complete the matching procedure with (20), (29) and (31) are

$$R_1(\zeta) = CP_1(\zeta)^{-\frac{1}{2}\gamma+\frac{3}{4}}(|P_1'(\zeta)|)^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}\zeta^2-2\int_0^\zeta(1-\tfrac{3}{2}\tau_0^2F_0'(\zeta_1))\frac{P_1(\zeta_1)}{P_1'(\zeta_1)}\,d\zeta_1\right] \tag{48}$$

and

$$Q_1(\zeta) = 2(\zeta-\zeta_0)\,R_1(\zeta)/P_1'(\zeta), \tag{49}$$

where C is a constant which may be determined from the matching process.

To satisfy the boundary conditions at the wall it is necessary to consider the other solution of (43), $P_2(\zeta)$, for which $P_2(0) = P_1(0)$. This is given by

$$P_2(\zeta) = \left[\int_{\zeta}^{\zeta_0} \tau_0^{\frac{1}{2}}[1-\tfrac{1}{2}\tau_0^2 F_0'(\zeta_1)]^{\frac{1}{2}}\,d\zeta_1 - c^{\frac{1}{2}} - 2\int_0^{\zeta_0} \tau_0^{\frac{1}{2}}[1-\tfrac{1}{2}\tau_0^2 F_0'(\zeta_1)]^{\frac{1}{2}}\,d\zeta_1\right]^2, \quad (50)$$

and associated with it are corresponding functions $Q_2(\zeta)$ and $R_2(\zeta)$. Thus we have a second solution of the form (41) available. To satisfy all the conditions at the wall a third solution is required, and for this we take a readily obtained solution of the linearized equations (13) and (14) of the form

$$\left.\begin{aligned} F_1(\zeta,\tau) &= F_0'(\zeta)\,L(\tau) + M(\tau), \quad \Theta_1(\zeta,\tau) = \Theta_0'(\zeta)\,L(\tau), \\ \text{with} \quad & 2\tau L' + L + \tau^3 M' = 0, \end{aligned}\right\} \quad (51)$$

where $L(\tau)$ is arbitrary. If we choose $L(\tau) = T^{\gamma+\frac{1}{2}}\exp[-P_1(0)/T]$ we may satisfy the three boundary conditions at the wall using the three solutions now available. Except at $\zeta = 0$ the latter two solutions are exponentially smaller than the original solution involving $P_1(\zeta)$, and so for $\zeta > 0$ are negligible in comparison.

We are now in a position to estimate the departure from the one-dimensional solution, close to $\tau = \tau_0$, of the displacement thickness, the skin friction and the heat transfer as predicted by our lowest eigenfunction. The principal contribution to the displacement thickness δ^* may be calculated from the outer-layer solution, and from (35) we obtain

$$\log \delta^* = -0{\cdot}656/(\tau-\tau_0) + 1{\cdot}452 \log(\tau-\tau_0) + O(1). \quad (52)$$

The dependence on τ of the logarithms of the perturbation skin friction and heat transfer near $\tau = \tau_0$ follows from (41) and is

$$\left.\begin{aligned} &-1{\cdot}360/(\tau-\tau_0) + 0{\cdot}952\log(\tau-\tau_0) + O(1), \\ &-1{\cdot}360/(\tau-\tau_0) - 0{\cdot}048\log(\tau-\tau_0) + O(1), \end{aligned}\right\} \quad (53)$$

respectively, since $P_1(0) = 1{\cdot}360$.

4. The final decay: $\tau \gg 1$

We conclude by investigating the departures from the steady-state solutions of (5) and (6) as $\tau \to \infty$. In other words we study the final decay in the evolution of the solution to its steady state. We write

$$f(\eta,\tau) = f_0(\eta) + f_1(\eta,\tau), \quad \theta(\eta,\tau) = \theta_0(\eta) + \theta_1(\eta,\tau), \quad (54)$$

where f_0 and θ_0 represent the steady-state solutions satisfying (5) and (6), and assume that $|f_1| \ll |f_0|$ and $|\theta_1| \ll |\theta_0|$. If we substitute (54) into (3) and neglect products of small quantities then we find that f_1 and θ_1 satisfy

$$\left.\begin{aligned} (1-\tfrac{1}{2}\tau f_0')\frac{\partial^2 f_1}{\partial\tau\,\partial\eta} + \tfrac{1}{2}\tau f_0''\frac{\partial f_1}{\partial\tau} - \tfrac{3}{4}f_0\frac{\partial^2 f_1}{\partial\eta^2} - \tfrac{3}{4}f_0'' f_1 + f_0'\frac{\partial f_1}{\partial\eta} &= \theta_1 + \frac{\partial^3 f_1}{\partial\eta^3}, \\ (1-\tfrac{1}{2}\tau f_0')\frac{\partial\theta_1}{\partial\tau} + \tfrac{1}{2}\tau\theta_0'\frac{\partial f_1}{\partial\tau} - \tfrac{3}{4}f_0\frac{\partial\theta_1}{\partial\eta} - \tfrac{3}{4}\theta_0' f_1 &= \frac{\partial^2\theta_1}{\partial\eta^2}, \end{aligned}\right\} \quad (55)$$

together with

$$\left.\begin{aligned} f_1 = \partial f_1/\partial\eta = \theta_1 = 0 \quad \text{on} \quad \eta = 0, \\ \theta_1, \partial f_1/\partial\eta \to 0 \quad \text{as} \quad \eta\to\infty \quad \text{and as} \quad \tau\to\infty. \end{aligned}\right\} \tag{56}$$

We now seek solutions of (55) and (56) in the form

$$\left.\begin{aligned} f_1(\eta,\tau) &= T^*(\tau)\{\bar{F}_0(\eta)+\tau^{-1}\bar{F}_1(\eta)+O(\tau^{-2})\}, \\ \theta_1(\eta,\tau) &= T^*(\tau)\{\bar{\Theta}_0(\eta)+\tau^{-1}\bar{\Theta}_1(\eta)+O(\tau^{-2})\}, \end{aligned}\right\} \tag{57}$$

where we assume that $T^*(\tau)$ is exponentially small as $\tau\to\infty$ but is otherwise, as yet, undetermined. Substituting (57) into (55) we find that the leading terms satisfy the equations

$$f_0'\bar{F}_0'-f_0''\bar{F}_0 = 0, \quad f_0'\bar{\Theta}_0-\theta_0'\bar{F}_0 = 0,$$

and these equations readily admit the solutions

$$\bar{F}_0 = \beta_0 f_0', \quad \bar{\Theta}_0 = \beta_0\theta_0'. \tag{58}$$

However, with $\bar{F}_0$ and $\bar{\Theta}_0$ determined in this way we see from (57) that f_1 and θ_1 satisfy the conditions as $\eta,\tau\to\infty$ but violate the conditions at $\eta = 0$. We therefore require an inner layer in which adjustment can take place. Now the boundary conditions show that θ_1 must have a point of inflexion and we note that, since θ_0' is monotonic, this must be in the inner region. If we assume that the solution structure in the inner region is determined by the highest derivatives in τ and η, and our discussion below based upon this assumption is self-consistent, then it follows that θ_1 possesses a point of inflexion when $\tau f_0' = 2$ which is equivalent to $\tau\eta = O(1)$ when $\tau \gg 1$. Accordingly we define a new variable

$$\bar{\eta} = \tau\eta, \tag{59}$$

and if $f_0 = \alpha\eta^2+O(\eta^3)$, $\theta_0 = 1+\beta\eta+O(\eta^2)$ as $\eta\to 0$, where α and β are known constants, then equations (55) become, with

$$f_1(\eta,\tau) = \bar{f}_1(\bar{\eta},\tau), \quad \theta_1(\eta,\tau) = \bar{\theta}_1(\bar{\eta},\tau),$$

$$\left.\begin{aligned} &(1-\alpha\bar{\eta})\left\{\tau\frac{\partial^2\bar{f}_1}{\partial\tau\,\partial\bar{\eta}}+\frac{\partial\bar{f}_1}{\partial\bar{\eta}}+\bar{\eta}\frac{\partial^2\bar{f}_1}{\partial\bar{\eta}^2}\right\}+\tau\alpha\frac{\partial\bar{f}_1}{\partial\tau}+3\alpha\bar{\eta}\frac{\partial\bar{f}_1}{\partial\bar{\eta}} \\ &\quad -\tfrac{3}{4}\alpha\bar{\eta}^2\frac{\partial^2\bar{f}_1}{\partial\bar{\eta}^2}-\frac{3\alpha}{2}\bar{f}_1-\bar{\theta}_1-\tau^3\frac{\partial^3\bar{f}_1}{\partial\bar{\eta}^3} = O\left\{\max\left(\frac{\partial\bar{f}_1}{\partial\tau},\ \tau^{-1}\bar{f}_1\right)\right\}, \\ &(1-\alpha\bar{\eta})\left\{\frac{\partial\bar{\theta}_1}{\partial\tau}+\frac{\bar{\eta}}{\tau}\frac{\partial\bar{\theta}_1}{\partial\bar{\eta}}\right\}+\tfrac{1}{2}\beta\tau\left\{\frac{\partial\bar{f}_1}{\partial\tau}+\frac{\bar{\eta}}{\tau}\frac{\partial\bar{f}_1}{\partial\bar{\eta}}\right\}-\frac{3}{4}\frac{\alpha^2\bar{\eta}^2}{\tau}\frac{\partial\bar{\theta}_1}{\partial\bar{\eta}} \\ &\quad -\tfrac{3}{4}\beta\bar{f}_1-\tau^2\frac{\partial^2\bar{\theta}_1}{\partial\bar{\eta}^2} = O\left\{\max\left(\tau^{-1}\frac{\partial\bar{\theta}_1}{\partial\tau},\ \frac{\partial\bar{f}_1}{\partial\tau}, \tau^{-2}\bar{\theta}_1,\ \tau^{-1}\bar{f}_1\right)\right\}. \end{aligned}\right\} \tag{60}$$

By using (58) expanded for small η and using (59), we see that in order to effect a match with the outer solution the solution in the inner region must exhibit the asymptotic behaviour as $\bar{\eta}\to\infty$

$$\bar{f}_1 \sim 2\alpha\beta_0 T^*(\tau)\,\bar{\eta}/\tau, \quad \bar{\theta}_1 \sim \beta\beta_0 T^*(\tau), \tag{61}$$

which suggests that in the inner region

$$\left.\begin{aligned} \bar{f}_1(\bar{\eta},\tau) &= T^*(\tau)\,\tau^{-1}\{\mathscr{F}_1(\bar{\eta})+\tau^{-1}\mathscr{F}_2(\bar{\eta})+O(\tau^{-2})\}, \\ \bar{\theta}_1(\bar{\eta},\tau) &= T^*(\tau)\{\bar{\phi}_0(\bar{\eta})+\tau^{-1}\bar{\phi}_1(\bar{\eta})+O(\tau^{-2})\}. \end{aligned}\right\} \tag{62}$$

Substituting for $\bar{f}_1$ and $\bar{\theta}_1$ in (60) we find that the balance of terms which is necessary if the inner and outer solutions are to match, and the diffusion terms are to play the role required of them in enabling us to satisfy the conditions at the wall, requires that T^* must satisfy

$$\frac{1}{T^*}\frac{dT^*}{d\tau} \sim a_1\tau^2 \quad \text{as} \quad \tau \to \infty, \tag{63}$$

where a_1 is a constant which will be determined later. With this behaviour for T^* we find that $\bar{\mathscr{F}}_1$ and $\bar{\phi}_0$ satisfy

$$\left.\begin{aligned}\bar{\mathscr{F}}_1''' - a_1(1-\alpha\bar{\eta})\bar{\mathscr{F}}_1' - a_1\alpha\bar{\mathscr{F}}_1 &= 0,\\ \bar{\phi}_0'' - a_1(1-\alpha\bar{\eta})\bar{\phi}_0 - \tfrac{1}{2}a_1\beta\bar{\mathscr{F}}_1 &= 0.\end{aligned}\right\} \tag{64}$$

Further considerations show that we may, in fact, without loss of generality take

$$\frac{1}{T^*}\frac{dT^*}{d\tau} = a_1\tau^2 + a_2\tau + a_3 + a_4/\tau; \tag{65}$$

thus

$$T^* = \tau^{a_4}\exp\{\tfrac{1}{3}a_1\tau^3 + \tfrac{1}{2}a_2\tau^2 + a_3\tau\}, \tag{66}$$

and the constants a_1, a_2, a_3 and a_4 remain to be determined. The first of equations (64), which satisfies $\bar{\mathscr{F}}_1(0) = \bar{\mathscr{F}}_1'(0) = 0$ together with the matching condition, represents an eigenvalue problem for the first of the unknown constants, a_1, in (66). This was anticipated from our discussion in §1. If we write

$$\xi = \lambda_1(\bar{\eta} - \alpha^{-1}), \quad \lambda_1^3 = -a_1\alpha, \quad \mathscr{F}_1(\xi) = \bar{\mathscr{F}}_1(\bar{\eta}), \tag{67}$$

then the first of equations (64) becomes

$$\mathscr{F}_1''' - \xi\mathscr{F}_1' + \mathscr{F}_1 = 0, \tag{68}$$

which gives

$$\mathscr{F}_1''(\xi) = C_1\,\mathrm{Ai}\,(\xi), \tag{69}$$

with $\mathscr{F}_1'''(\xi_0) = 0$ where $\xi_0 = -\lambda_1/\alpha$, and so for a non-trivial solution ξ_0 must satisfy $\mathrm{Ai}'(\xi) = 0$. The smallest value of $|\xi|$ which satisfies this equation yields the dominant eigenvalue and this is $\xi_0 = -1{\cdot}019$, which in turn gives $a_1 = -1{\cdot}058\alpha^2$. The remaining constant C_1 in (69) is related to the constant β_0 in the outer solution (58) by the matching criterion. We obtain as $\bar{\eta} \to \infty$

$$\bar{\mathscr{F}}_1(\bar{\eta}) \sim 0{\cdot}809C_1\lambda_1\bar{\eta}, \tag{70}$$

and consequently, from (61), (62) and (70) we determine the relationship between β_0 and C_1 as $\alpha\beta_0 = 0{\cdot}404C_1\lambda_1$. The solution of (64) for $\bar{\phi}_0$ may now be determined. The solution which satisfies the condition $\bar{\phi}_0(0) = 0$ and the matching requirement is simply

$$\bar{\phi}_0(\bar{\eta}) = \tfrac{1}{2}(\beta/\alpha)\bar{\mathscr{F}}_1'(\bar{\eta}). \tag{71}$$

In principle a study of the higher order terms in (62) may now be made which will yield values for the remaining constants in (66). In practice the analysis is formidable, although one further term has been considered by Andrews (1969), who finds that $a_2 = -0{\cdot}9268$.

In conclusion, we note that in this final decay to the steady state the perturbation displacement thickness, skin friction and heat transfer are all dominated by the term $\exp(\frac{1}{3}a_1\tau^3)$, showing that as $\tau \to \infty$ the steady-state solution is rapidly achieved.

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